Convergence of the Mayer Series via Cauchy Majorant Method with Application to the Yukawa Gas in the Region of Collapse

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Abstract

We construct majorant functions $\Phi(t, z)$ for the Mayer series of pressure satisfying a nonlinear differential equation of first order which can be solved by the method of characteristics. The domain $|z| < (e\tau)^{-1}$ of convergence of Mayer series is given by the envelop of characteristic intersections. For non negative potentials we derive an explicit solution in terms of the Lambert W–function which is related to the exponential generating function T of rooted trees as $T(x) = -W(-x)$. For stable potentials the solution is majorized by a non negative potential solution. There are many choices in this case and we combine this freedom together with a Lagrange multiplier to examine the Yukawa gas in the region of collapse. We give, in this paper, a sufficient condition to establish a conjecture of Benfatto, Gallavotti and Nicoló. For any $\beta \in [4\pi, 8\pi)$, the Mayer series with the leading terms of the expansion omitted (how many depending on β) is shown to be convergent provided an improved stability condition holds. Numerical calculations presented indicate this condition is satisfied if few particles are involved.

1 Introduction

One of main investigations in equilibrium statistical mechanics concerning the determination of phase diagrams is to define regions of the parameter space in which a phase transition does not occur. According to Lee–Yang Theorem [YL, LY] theses regions are free of zeros of the partition function which, for lattice models with attractive pair potentials, are located in the unit circle of the complex activity plane. Another way of determining whether the phase diagram is free of singularities is given by Mayer series. Although the theory based on these series does not describe

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the equation of state beyond the condensate point, it is able, however, to deal with continuum as well as lattice space for a larger class of admissible pair potentials.

The domain of convergence of Mayer series for classical gas with stable two–body summable potential can be established using various different techniques (see Ruelle [R], Brydges [B] and references therein). There are cases, including attractive potential at short distances and Yukawa gas at inverse temperature β up to 4π, which require the use of finitely [GM] or infinitely [Be, I] many Mayer series iterations. The present investigation uses the flow equation of Ursell functions introduced by Brydges and Kennedy [BK, BK1] in order to obtain previously known and some new results on the convergence of Mayer series using methods of partial differential equations (PDE).

The radius of convergence of a Mayer series is often determined by a "non–Physical" singularity in the complex activity z–plane. In the present article we shall write a majorant series $\Phi(t,z)$ for the pressure $p(\beta,z)$ of a gas which satisfies a nonlinear partial differential equation of first order. The "non–Physical" singularity can be recognized as the location, in the phase space $\Omega = \{(t,z)\in\mathbb{R}^2 : t\in[t_0,t_1]\,,\ z\geq 0\}$, of characteristics intersection (t is eventually related with β). As in Tonks' model of one–dimensional rods with hard core interaction, the singularity of $\Phi(t,z)$ is manifested by the W–Lambert function, an indication that this singularity is of combinatorial nature [K].

In the application to the Yukawa gas, the variable $t\in[t_0,0]$ parameterizes the decomposition of the Yukawa potential into scales, with t_0 being related with the short distance cutoff, which is removed if $t_0\to-\infty$. There is a conjecture that the radius of convergence of the corresponding Mayer series remains finite as $t_0\to-\infty$ for $\beta\in[4\pi,8\pi)$ if one subtracts the leading terms of the expansion, how many depending on β. Stated as an open problem in [Be] and previously formulated in [BGN, GN], the conjecture will be referred here as BGN's Conjecture. Brydges–Kennedy [BK] have proved convergence of Mayer series for $4\pi\leq\beta<16\pi/3$ with the $O\left(z^2\right)$ term omitted and shown how it could be extended up to $\beta<6\pi$ if an improved stability condition for three particles holds. In the present work we establish BGN's conjecture up to $\beta<8\pi$, assuming a mild improvement on the stability condition for the energy of non–neutral n–particle configurations. As our calculation indicates, this condition can be easily verified numerically if n is not too large since the minimal value attained by the energy maintains a comfortable distance from the lower bound needed to establish convergence.

At inverse temperature $8\pi\left(1+(2r+1)^{-1}\right)^{-1}$, $r=1,2,\ldots$, neutral multipoles of order $\leq 2r$ become unstable and collapse if the cutoff is removed. Even coefficients of the Mayer series up to order $2r$ diverges as $t_0\to-\infty$ and they remain finite otherwise because there occur cancellations when unstable multipoles interact with another particle. Our proof of convergence has two distinct parts. In the first, the solution $\Phi^{(k)}(t,z)$ of a nonlinear first order PDE modified by Lagrange multipliers, with $\Phi^{(k)}(t_0,z)=z$, is shown to be analytic in a disc whose radius remains positive as $t_0\to-\infty$ for any $\beta<\beta_k=8\pi\left(1+k^{-1}\right)^{-1}$. In the second part, $\Phi^{(k)}$ is shown to majorize the Mayer series with even coefficients up to order k subtracted. The two ingredients here are an improved stability bound for non–neutral configurations and translational invariance. All estimates are done in the infinite volume limit to avoid boundary problems. As the Yukawa potential decays exponentially fast this choice, although not relevant from the point of view of thermodynamics, is suitable to exhibit the necessary cancellations right at the beginning.

This paper is organized as follows. Section 2 contains a review of Gaussian integrals, including their relation to the partition function of classical gases, and the statements of our results in

Theorems 2.2 and 2.3. Section 3 gives an alternative derivation of Ursell integral equation of Brydges–Kennedy. The integral equation is the starting point for the Majorant differential equation presented in Section 4. This first order nonlinear partial differential equation is solved in Section 5 by the method of characteristics for non negative and stable potentials, concluding the proof of Theorem 2.2. Section 6 and 7 give an application of the previous section to the ultraviolet problem of the two–dimension Yukawa gas. Section 6 reestablishes previously known results via our method. In Sections 7 we prove BGN's Conjecture under the assumption of an improved stability condition (Theorem 2.3) which has been verified numerically for $n = 3, 4$ and 5 (see Remark 7.3).

2 Gaussian Integrals and Summary of Results

To fix our notations, we need some facts on Gaussian integrals and their relation to the grand partition function (see references [FS, Br, BM], for details).

The state of a particle $\xi = (x, \sigma)$ consists of its position x and some internal degrees of freedom σ. *Examples:* 1. For one–component lattice gases, $\xi = x \in \mathbb{Z}^d$; 2. For a two–component continuous Coulomb gas, $\xi = (x, \pm e)$ where $x \in \mathbb{R}^d$ and e is the electrical charge; 3. For dipole lattice gas, $\xi = (x, \pm e_j)$ where $x \in \mathbb{Z}^d$ and e_j is the canonical unit vector.

The particles are constrained to move in a finite subset X of $\mathbb{Z}^d$ (or a compact region of $\mathbb{R}^d$). The lattice is preferred for pedagogical reason but we shall also consider particles in a continuum space. S is in general a compact space and we set $\Lambda = X \times S$. By the triple $(\Omega, \Sigma, d\rho)$ we mean the measure space of a single particle with $\Omega = \mathbb{R}^d \times S$, Σ a sigma algebra generated by the cylinder sets of $\mathbb{R}^d \times S$ and $d\rho(\xi) = d\mu(x)\, d\nu(\sigma)$ a product of positive measures ($d\mu(x) = \sum_{n \in \mathbb{Z}^d} \delta(x-n)\, d^d x$ is a counting or Lebesgue $d\mu(x) = d^d x$ measure times a probability measure $d\nu$ on S).

The grand canonical partition function is given by

$$\Xi_\Lambda(\beta, z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{\Lambda^n} d^n\rho(\boldsymbol{\xi})\, \psi_n(\boldsymbol{\xi}) \tag{2.1}$$

where, for each state $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$ of n–particles with activity z,

$$\psi_n(\boldsymbol{\xi}) = e^{-\beta U_n(\boldsymbol{\xi})} \tag{2.2}$$

is the associate Boltzmann factor,

$$U_n(\boldsymbol{\xi}) = \sum_{i<j} \vartheta(\xi_i, \xi_j) \tag{2.3}$$

the interaction energy and β is the inverse temperature. The two–body potential $\vartheta : \Lambda \times \Lambda \longrightarrow \mathbb{R}$ is a jointly measurable function and we use, sometimes, the abbreviation $\vartheta_{ij} := \vartheta(\xi_i, \xi_j)$. The finite volume pressure $p_\Lambda(\beta, z)$ of a classical gas is defined by

$$p_\Lambda(\beta, z) = \frac{1}{\beta\, |\Lambda|} \ln \Xi_\Lambda(\beta, z)\,. \tag{2.4}$$

Without loss, ϑ is defined in the whole space $\mathbb{Z}^d$ (or $\mathbb{R}^d$) - the Green's function of some differential equation with free (insulating) boundary conditions - and it is assumed to be of the form

$$\beta\vartheta(\xi, \xi') = \sigma\sigma' v\,(x - x') \tag{2.5}$$

for some real–valued measurable functions v satisfying:

1. *Regularity at origin* $|v(0)| \leq 2B < \infty$;
2. *Summability*
$$\|v\| := \int d\mu(x)\ |v(x)| < \infty\ ; \tag{2.6}$$
3. *Positivity*
$$\sum_{1\leq i,j\leq n} \overline{z}_i\, z_j\, v\,(x_i - x_j) \geq 0\,, \tag{2.7}$$
for any numbers $x_1, \ldots, x_n \in \mathbb{R}^d$ and $z_1, \ldots, z_n \in \mathbb{C}$, $n = 1, 2, \ldots$.

Note these assumptions ensure that ϑ is a stable potential, i.e., choosing $z_1 = \cdots = z_n = 1$ in (2.7),
$$\beta U_n(\boldsymbol{\xi}) \geq -\frac{1}{2}\sum_{1\leq i\leq n} \sigma_i^2\,|v(0)| \geq -Bn \tag{2.8}$$
holds with $B \geq 0$ independent of n (see Section 3.2 of Ruelle [R], for further comments).

Since v is a positive definite symmetric function, it defines a Gaussian measure μ_v with mean $\displaystyle\int d\mu_v(\phi)\phi(x)\ = 0$ and covariance $\int d\mu_v(\phi)\,\phi(x)\,\phi(x') = v(x - x')$. Introducing the inner product
$$(\phi, \eta) = \int d\mu(x)\,\phi(x)\,\eta(x)$$
and the indicator function
$$\eta_n(\cdot) = \eta(\boldsymbol{\xi}; \cdot) = \sum_{j=1}^{n} \sigma_j\ \delta\,(\cdot - x_j) \tag{2.9}$$
of a n–state $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$, we have
$$\psi_n(\boldsymbol{\xi}) = \int d\mu_v\,(\phi)\ :e^{i(\phi,\eta_n)}:_v \tag{2.10}$$
where $::_v$ is the normal ordering with respect to the covariance v:
$$:e^{i\sigma\phi(x)}:_v = \exp\left\{\frac{\sigma^2}{2}v\,(0)\right\}\ e^{i\sigma\phi(x)}\,. \tag{2.11}$$

Substituting (2.10) into (2.1) yields
$$\Xi_\Lambda(\beta, z) = \int d\mu_v\,(\phi)\ \exp\left\{V_0(\phi)\right\} \tag{2.12}$$
where
$$V_0(\phi) = z\int_\Lambda d\rho(\xi)\ :e^{i\sigma\phi(x)}:_v\ . \tag{2.13}$$

Example 2.1 *1.* $V_0(\phi) = z \sum_{x\in X} : e^{i\phi(x)} :_v$ *for one–component lattice gas;*

2. $V_0(\phi) = z \int_X d^d x : \cos\phi(x) :_v$ *for standard Coulomb or Yukawa gas;*

3. $V_0(\phi) = z \sum_{x\in X} \int_{|\sigma|=1} d^d\sigma : \cos(\sigma\cdot\nabla\phi(x)) :_v$ *for dipole lattice gas.*

In Example 2 we have respectively, $v = (-\Delta)^{-1}$ and $v = (-\Delta+1)^{-1}$, the fundamental solution of Laplace and Yukawa equations. The potential of two dipoles, of moment σ and σ', in Example 3. is given by $v = (\sigma\cdot\nabla)(\sigma'\cdot\nabla')(-\Delta)^{-1}$, second difference of the fundamental solution of discrete Laplace equation. V_0 in Examples 2 and 3 contain cosine function instead exponential. This form expresses the pseudo–charge symmetry required for the existence of thermodynamic limit $\lim_{\Lambda\nearrow\mathbb{R}^d(\mathbb{Z}^d)} p_\Lambda = p$ of respective systems (see, e.g., [S]).

The potentials in Examples 2 and 3 do not fulfill all requirements: Coulomb and dipole potentials do not satisfy summability whereas the Yukawa potential fails to be regular at origin. To circumvent these problems, a one parameter smooth decomposition of covariance $v(t,x)$ may be introduced so that $v(t_0,x) = 0$ and $\int_{t_0}^{t_1} \|\dot v(s,\cdot)\|\, ds < \infty$ holds for any finite interval $[t_0,t_1]$. This allows us to approach the ultraviolet problem of Yukawa gas by letting $t_0 \to -\infty$ maintaining t_1 fixed (e.g. $t_1 = 0$) in Example 2, and the infrared problem of dipole gas in Example 3 as the limit $t_1 \to \infty$ with $t_0 = 0$ fixed. The application in the present work deals only with the former problem.

In all examples $V_0(\phi)$ is a bounded perturbation to the Gaussian measure μ_v satisfying $|V_0(\phi)| \le z\,|\Lambda|$. Brydges and Kennedy [BK] have used the so called sine–Gordon representation (2.12) to derive a PDE in the context of Mayer expansion. Although based in their work, our PDE approach in the present article is remarkably different.

It is convenient to write a flow

$$V_\Lambda(t,z;\phi) = \ln\left(\mu_{v(t)} * \exp\{V_0(\phi)\}\right) \tag{2.14}$$

starting from $V_\Lambda(0,z;\phi) = V_0(\phi)$ where $\mu_v * F$ denotes convolution of F by the Gaussian measure μ_v which, by the Wick theorem, is given for the one–component lattice case by

$$\int d\mu_v(\zeta)\, F(\phi+\zeta) = \exp\left\{\frac{1}{2}\sum_{x,y\in\mathbb{Z}^d} v(t,x-y)\,\frac{\partial^2}{\partial\phi(x)\partial\phi(y)}\right\} F(\phi)\,. \tag{2.15}$$

To derive the standard Mayer expansion, the covariance $v(t)$ is chosen to be any $\mathscr{C}^1$ parameterization such that $v(t_0) = 0$ and $v(t_1) = v$. The parameterization involving multiscales, $\dot v(t)$ is required to be of the form (2.5) satisfying properties $1-3$ with $\dot B = \dot B(t) < \infty$ for all $t \in [t_0,t_1]$. $v(t)$ is said to be admissible if fulfills all these properties. Note that $V_0(\phi)$ also depends on t through the normal ordering $: :_{v(t)}$.

Equation (2.14) can be written as

$$e^{V_\Lambda(\beta,z,\phi)} = \Xi_\Lambda(\beta, ze^{i\boldsymbol{\phi}}) \tag{2.16}$$

where $\Xi_\Lambda(\beta, \boldsymbol{w})$ generalizes the grand partition function by including a complex valued activity $w = w(\xi)$ which depends on the particle state:

$$\Xi_\Lambda(\beta, \boldsymbol{w}) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} d^n \rho(\boldsymbol{\xi})\, w_n(\boldsymbol{\xi})\, \psi_n(\boldsymbol{\xi}) \tag{2.17}$$

with $w_n(\boldsymbol{\xi}) = \prod_{j=1}^{n} w(\xi_j)$ and (2.1) is recovered if $w(\xi) = z$.

We also have

$$\beta \left| p_\Lambda(\beta, z) \right| = \frac{1}{|\Lambda|} \left| V_\Lambda\left(t_1, z, 0\right) \right| \leq \sup_\phi \frac{1}{|\Lambda|} \left| V_\Lambda\left(t_1, z, \phi\right) \right| \tag{2.18}$$

and, as t goes to t_0 (the infinite temperature limit),

$$\lim_{t \to t_0} V_\Lambda\left(t, z, 0\right) = V_0(0) = z\left|\Lambda\right| \quad .$$

The flow $\{V_\Lambda(t, z, \phi)/\left|\Lambda\right| : 0 \leq t \leq 1\}$ provides us an upper bound for the pressure of gases described above, starting from the pressure of an ideal gas.

The results presented in this article can be summarized as follows.

Theorem 2.2 *Let $\Phi = \Phi(t, z)$ be the classical solution of*

$$\Phi_t = \frac{1}{2}\Gamma\left(z\Phi_z\right)^2 + \dot{B}\left(z\Phi_z - z\right) \tag{2.19}$$

defined in $t_0 < t \leq t_1$, $z \geq 0$ with $\Phi(t_0, z) = z$ where $\Gamma = \Gamma(t) = \|\dot{v}(t)\|$, $B = B(t)$ is defined by (2.8) with v replaced by $v(t)$. Then

$$\beta p_\Lambda(\beta, z) \leq \Phi(t_1, z)$$

holds uniformly in Λ. The solution Φ has a power series

$$\Phi(t, z) = z + \sum_{n=2}^{\infty} A_n(t)\; z^n$$

with $A_n(t) \geq 0$, convergent in the open disc $D = \left\{|z| < (e\tau)^{-1}\right\}$ where

$$\tau(t_0, t) = \inf\left(\int_{t_0}^{t} \Gamma(s) \exp\left\{2\int_s^t \dot{B}(s')\; ds'\right\} ds,\;\; \exp\left\{\int_{t_0}^{t} \dot{B}(s')\; ds'\right\} \int_{t_0}^{t} \Gamma(s) ds\right)$$

and is majorized by

$$\Phi(t, z) \leq \frac{-1}{\tau}\left(W\left(-z\tau\right) + \frac{1}{2}W^2\left(-z\tau\right)\right) \tag{2.20}$$

where $W(x)$ is the Lambert W–function defined as the inverse of $f(W) = We^W$.

Theorem 2.3 *Given an odd number $k \in \mathbb{N}$, suppose an improved stability condition*

$$\dot{U}_n(t;\boldsymbol{\xi}) \geq -(n-\delta_n)\dot{B}(t) \tag{2.21}$$

holds for all n–particle states $\boldsymbol{\xi} = (\boldsymbol{\sigma}, \boldsymbol{x})$ such that $\sum_{i=1}^{n} \sigma_i \neq 0$ with

$$\delta_n > \frac{1}{n}$$

and $n \leq k$. Then, the pressure $p(\beta, z)$ of Yukawa gas, with even Mayer coefficients up to order k subtracted, exists and has a convergent power series provided

$$\beta < \beta_k \qquad \text{and} \qquad |z| < \frac{\beta_k - \beta}{\beta_k \beta e}$$

where $\beta_k = 8\pi/(1+k^{-1})$ is the k–th threshold.

Theorems 2.2 and 2.3 will be proven in Sections 5 and 7, respectively, by majorizing the solution of (2.19), possibly modified by a Lagrange multiplier L, by an explicitly solution of (2.19) with $B = 0$ and Γ replaced by Γ/f for a conveniently chosen function f .

3 Ursell Flow Equation

The power series of the pressure (2.4) in the activity z is called Mayer series:

$$\beta p_\Lambda(\beta, z) = \sum_{n=1}^{\infty} b_{\Lambda,n}\, z^n \quad . \tag{3.1}$$

The Mayer coefficient $b_{\Lambda,n}$ has an expression analogous to (2.1),

$$|\Lambda|\; b_{\Lambda,n} = \frac{1}{n!} \int_{\Lambda^n} d^n\rho(\boldsymbol{\xi})\, \psi_n^c(\boldsymbol{\xi}) \tag{3.2}$$

where, for each state $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n) \in (\mathbb{R} \times S)^n$ of n–particles, $\psi_n^c(\boldsymbol{\xi})$ is the Ursell functions of order n. In terms of Mayer graphs,

$$\psi_n^c(\boldsymbol{\xi}) = \sum_G \prod_{(i,j)\in G} \left(e^{-\beta\vartheta_{ij}} - 1\right) \tag{3.3}$$

where G runs over all connected graphs on $I_n = \{1, \ldots, n\}$ (G is connected if for any two vertex k, $l \in I_n$ there is a path of edges $(i,j) \in I_n \times I_n$ in G connecting k to l).

Equation (3.3) is not useful to investigate whether (3.1) is a convergent series since $|\sum_G 1| > c\,(n!)^{1+\delta}$ holds for some $c, \delta > 0$[1]. We shall instead derive a flow equation for $\psi_n^c(\boldsymbol{\xi})$ that incorporate the cancellations required to reduce the cardinality of the sum in (3.3) to $O(n!)$.

[1]The number of Mayer graphs with n vertices is $2^{n(n-1)/2}$ and the number of connected Mayer graphs is of the same order. For this, note that a tree, the minimal connected graph, has exactly $n-1$ bonds, remaining $\frac{n^2}{2} - \frac{3n}{2} + 1$ bonds to be freely chosen to be connected.

Given $\boldsymbol{\xi} = (\xi_j)_{j\geq 1} \in (\mathbb{R}\times S)^\infty$, let $P_n\boldsymbol{\xi} = (\xi_1,\ldots,\xi_n)$ denote the projection of $\boldsymbol{\xi}$ into the subspace $(\mathbb{R}\times S)^n$ generated by its n first components. If $I\subset\mathbb{N}$ is a finite set $I=\{i_1,\ldots,i_n\}$, we set $P_I\boldsymbol{\xi} = (\xi_{i_1},\ldots,\xi_{i_n})$. To each sequence $\boldsymbol{\xi}$ we associate a complex Boltzmann function

$$u(t,\boldsymbol{\xi}) = \exp\left\{-\sum_{i<j}\sigma_i\sigma_j v(t,x_i-x_j) + i\sum_j \sigma_j\phi(x_j)\right\}$$

and define

$$u_n(t,\boldsymbol{\xi}) = u(t,P_n\boldsymbol{\xi}) = e^{-\beta U_n(\boldsymbol{\xi})+i(\phi,\eta_n)}\,, \tag{3.4}$$

for $n\geq 2$ where $U_n(\boldsymbol{\xi}) = U(P_n\boldsymbol{\xi})$ is the interaction energy (2.3) and $\eta_n(\cdot) = \eta(P_n\boldsymbol{\xi},\cdot)$ is the indicator function (2.9). Note that $u_n : \mathbb{R}_+\times(\mathbb{R}\times S)^n \longrightarrow \mathbb{C}$ is symmetric with respect to the permutation π of the index set $\{1,\ldots,n\}$:

$$u(t,P_n\boldsymbol{\xi}) = u(t,M_\pi P_n\boldsymbol{\xi})$$

where $M_\pi : (\xi_1,\ldots,\xi_n)\longrightarrow(\xi_{\pi_1},\ldots,\xi_{\pi_n})$ is the permutation matrix corresponding to π. Note also that $u_n(t,\boldsymbol{\xi})$ can be written as

$$u_n(t,\boldsymbol{\xi}) = \psi_n(\boldsymbol{\xi})\,e^{i(\phi,\eta_n)} = \mu_{v(t)}* : e^{i(\phi,\eta_n)} :_{v(t)}, \tag{3.5}$$

where ψ_n is the Boltzmann function (2.2), with $v=v(t)$ being an admissible covariance.

Now, let $\boldsymbol{u}(t) = (u_n(t,\boldsymbol{\xi}))_{n\geq 0}$ denote the sequence of these functions with $u_0(t)=1$ and $u_1(t,\boldsymbol{\xi}) = e^{i\sigma_1\phi(x_1)}$. At $t=t_0$, $v(t_0)=0$ (absence of interactions) and $\boldsymbol{u}(t_0) = \boldsymbol{u}^{(0)}$ is the sequence with $u_n^{(0)}(\boldsymbol{\xi}) = e^{i(\phi,\eta_n)}$ for $n\geq 1$ and $u_0^{(0)}(\boldsymbol{\xi}) = 1$. Using (3.5), (2.11) and (2.15), a simple computation yields

Proposition 3.1 *The sequence $\boldsymbol{u}(t) = (u_n(t,\boldsymbol{\xi}))_{n\geq 0}$ is the formal solution of the heat equation*

$$\boldsymbol{u}_t = \frac{1}{2}\Delta_{\dot v(t)}\boldsymbol{u} \tag{3.6}$$

with initial condition $\boldsymbol{u}(t_0) = \boldsymbol{u}^{(0)}$, where the action of Δ_D on $\boldsymbol{u}$ is a sequence $\Delta_D\boldsymbol{u}$ with

$$(\Delta_D u)_n = \sum_{1\leq i,j\leq n, i\neq j} D(x_i,x_j)\,\frac{\partial^2 u_n}{\partial\phi(x_i)\partial\phi(x_j)}\,. \tag{3.7}$$

Remark 3.2 1. *The derivatives in (3.7) are partial derivatives with respect to ϕ at the value x, even for continuous space. This has to be contrasted with the functional derivative in (2.15) whose correspondence to partial derivative can only be made for lattice space.*

2. *To emphasize the order of operation, we write $\Delta_D = \nabla\cdot D\nabla$ for the differential operator (3.7) if D is a off–diagonal symmetric matrix.*

Let $\mathfrak{S}$ denote the vector space over $\mathbb{C}$ of sequences $\boldsymbol{\psi} = (\psi_n(\boldsymbol{\xi}))_{n\geq 0}$ of bounded Lebesgue measurable functions $\psi_n(\boldsymbol{\xi}) = \psi(P_n\,\boldsymbol{\xi})$ on $(\mathbb{R}\times\mathbb{S})^n$ which are symmetric with respect to the permutation π of the index set $\{1,\ldots,n\}$, with $\psi_0(\boldsymbol{\xi}) = \psi\,(P_\emptyset\boldsymbol{\xi})\in\mathbb{C}$. We define a product operation $\circ : \mathfrak{S}\times\mathfrak{S}\longrightarrow\mathfrak{S}$ for any two sequences of symmetric functions $\boldsymbol{\psi}_1$, $\boldsymbol{\psi}_2$ as a sequence

$$\boldsymbol{\psi}_1\circ\boldsymbol{\psi}_2 = \boldsymbol{\psi} = (\psi_n(\boldsymbol{\xi}))_n \tag{3.8}$$

whose components are given by

$$\psi_n(\boldsymbol{\xi}) = \sum_{I\subseteq\{1,\ldots,n\}} \psi_1(P_I\boldsymbol{\xi})\,\psi_2(P_{I^c}\boldsymbol{\xi})$$

where $I^c = \{1,\ldots,n\}\setminus I$. Note that this operation is well defined and preserves the symmetry. $\mathfrak{S}$ endowed with the operation $\circ$ form a commutative algebra with the identity $\mathbf{1} = (\delta_n(\boldsymbol{\xi}))_{n\geq 0}$ given by

$$\delta(P_I\boldsymbol{\xi}) = \begin{cases} 1 & \text{if} \quad I = \emptyset \\ 0 & \text{if} \quad I \neq \emptyset \end{cases} .$$

Let $\mathfrak{S}_0$ and $\mathfrak{S}_1$ denote the subspace and the affine space of sequences $\boldsymbol{\psi} = (\psi_n(\boldsymbol{\xi}))_{n\geq 0}$ such that $\psi_0(\boldsymbol{\xi}) = 0$ and $\psi_0(\boldsymbol{\xi}) = 1$, respectively. We define the algebraic exponential of sequences $\boldsymbol{\phi} = (\phi_n(\boldsymbol{\xi}))_{n\geq 0}$

$$\mathscr{E}xp : \mathfrak{S}_0 \longrightarrow \mathfrak{S}_1$$

as the following formal series

$$\mathscr{E}xp\,(\boldsymbol{\phi}) = \mathbf{1} + \boldsymbol{\phi} + \frac{1}{2}\boldsymbol{\phi}\circ\boldsymbol{\phi} + \cdots + \frac{1}{n!}\underbrace{\boldsymbol{\phi}\circ\cdots\circ\boldsymbol{\phi}}_{n-\text{times}} + \cdots$$

and the algebraic logarithmic of sequences $\boldsymbol{\psi} = (\psi_n(\boldsymbol{\xi}))_{n\geq 0}$

$$\mathscr{L}n : \mathfrak{S}_1 \longrightarrow \mathfrak{S}_0$$

as $(\boldsymbol{\psi} \in \mathfrak{S}_0)$

$$\mathscr{L}n\,(\mathbf{1} + \boldsymbol{\psi}) = \boldsymbol{\psi} - \frac{1}{2}\boldsymbol{\psi}\circ\boldsymbol{\psi} + \cdots + \frac{(-1)^{n-1}}{n}\underbrace{\boldsymbol{\psi}\circ\cdots\circ\boldsymbol{\psi}}_{n-\text{times}} + \cdots .$$

Note that the following

$$\mathscr{L}n\,(\mathscr{E}xp\,(\boldsymbol{\phi})) = \boldsymbol{\phi} \qquad \text{and} \qquad \mathscr{E}xp\,(\mathscr{L}n\,(\mathbf{1} + \boldsymbol{\psi})) = \mathbf{1} + \boldsymbol{\psi}$$

holds for any $\boldsymbol{\phi}$, $\boldsymbol{\psi} \in \mathfrak{S}_0$, so $\mathscr{E}xp$ and $\mathscr{L}n$ are algebraic inverse function of each other. We thus have (for details, see Ruelle [R])

Proposition 3.3 *The sequence $\boldsymbol{\psi} = (\psi_n)_{n\geq 0}$ of Boltzmann functions given by (2.2) $n \geq 2$ with $\psi_0 = \psi_1(\boldsymbol{\xi}) = 1$, is an algebraic exponential of the sequence $\boldsymbol{\psi}^c = (\psi_n^c)_{n\geq 0}$ of Ursell functions given by (3.3) with $\psi_0^c = 0$ and $\psi_1^c(\boldsymbol{\xi}) = 1$:*

$$\boldsymbol{\psi} = \mathscr{E}xp\,(\boldsymbol{\psi}^c) . \tag{3.9}$$

Proof. Let $\langle\cdot,\cdot\rangle : \mathfrak{S}\times\mathfrak{S} \longrightarrow \mathbb{C}$ be the inner product

$$\langle\boldsymbol{\phi},\boldsymbol{\psi}\rangle = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int d^n\rho(\boldsymbol{\xi})\,\phi_n(\boldsymbol{\xi})\,\psi_n(\boldsymbol{\xi}) \tag{3.10}$$

and let $\boldsymbol{\chi} = (\chi_n(\boldsymbol{\xi}))_{n\geq 0}$ be the sequence of characteristic function of the set Λ^n:

$$\chi_n(\boldsymbol{\xi}) = \prod_{j=1}^{n} \chi_\Lambda(\xi_j)$$

with

$$\chi_\Lambda(\xi) = \begin{cases} 1 & \text{if } \xi \in \Lambda \\ 0 & \text{otherwise} \end{cases} .$$

Using

$$\langle \boldsymbol{\chi}, \boldsymbol{\phi} \circ \boldsymbol{\psi} \rangle = \langle \boldsymbol{\chi}, \boldsymbol{\phi} \rangle \, \langle \boldsymbol{\chi}, \boldsymbol{\psi} \rangle$$

($\boldsymbol{\phi} \longrightarrow \langle \boldsymbol{\chi}, \boldsymbol{\phi} \rangle = \langle \boldsymbol{\chi}, \boldsymbol{\phi} \rangle(z)$ defines an homeomorphism of $\mathfrak{S}$ into the algebra of formal power series) together with (3.9), (3.10) and (2.1), we have

$$\Xi_\Lambda(\beta, z) = \langle \boldsymbol{\chi}, \boldsymbol{\psi} \rangle(z) = \langle \boldsymbol{\chi}, \mathscr{E}xp\,(\boldsymbol{\psi}^c) \rangle(z) = \exp\left\{ \langle \boldsymbol{\chi}, \boldsymbol{\psi}^c \rangle(z) \right\}$$

which, in view of (3.1)–(3.3), proves the proposition.

$\square$

Replacing $\boldsymbol{\psi}$ in (3.9) by the complex Boltzmann sequence $\boldsymbol{u}(t) = (u_n(t, \boldsymbol{\xi}))_{n\geq 0}$, leads to the complex Ursell sequence $\boldsymbol{y}(t) = (y_n(t, \boldsymbol{\xi}))_{n\geq 0}$ given by

$$\boldsymbol{y}(t) = \mathscr{L}n\,(\boldsymbol{u}(t)) \quad . \tag{3.11}$$

Note that, in view of (3.11), (3.5), (3.9) and the fact

$$(u \circ u)_n\,(t, \boldsymbol{\xi}) = \sum_{I \subseteq \{1,\ldots,n\}} \psi(t, P_I \boldsymbol{\xi})\, \psi(t, P_{I^c}\, \boldsymbol{\xi}) e^{i(\phi, \eta_n)} = (\psi \circ \psi)_n\,(t, \boldsymbol{\xi}) e^{i(\phi, \eta_n)} \;,$$

we have $y_n(t, \boldsymbol{\xi}) = \psi_n^c(t, \boldsymbol{\xi})\, e^{i(\phi, \eta_n)}$. We shall use (3.11) together with (3.6) to derive a partial differential equation for the complex Ursell functions $y_n(t, \boldsymbol{\xi})$. A differential equation for the Ursell function $\psi_n^c(t, \boldsymbol{\xi})$, given by (3.3) with $\beta\vartheta\,(\xi_i, \xi_j)$ replaced by $\sigma_i \sigma_j v(t, x_i - x_j)$, is obtained setting $\phi = 0$ at the end of computations.

The derivative of (3.11) together with Remark 3.2.2, yields

$$\begin{aligned} \boldsymbol{y}_t &= \frac{1}{\boldsymbol{u}} \circ \boldsymbol{u}_t \\ &= \frac{1}{2} \mathscr{E}xp\,(-\boldsymbol{y}) \circ (\nabla \cdot \dot{v} \nabla \mathscr{E}xp\,(\boldsymbol{y})) \\ &= \frac{1}{2} \Delta_{\dot{v}} \boldsymbol{y} + \frac{1}{2} \nabla \boldsymbol{y} \bullet \dot{v} \nabla \boldsymbol{y} \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} (\nabla \boldsymbol{y} \bullet D \nabla \boldsymbol{y})_n &= \sum_{1 \leq i,j \leq n, i \neq j} D(x_i, x_j) \left(\frac{\partial \boldsymbol{y}}{\partial \phi(x_i)} \circ \frac{\partial \boldsymbol{y}}{\partial \phi(x_j)} \right)_n \\ &= \sum_{I \subset \{1,\ldots,n\}} \nabla y\,(P_I\, \boldsymbol{\xi}) \cdot D \nabla y\,(P_{I^c} \boldsymbol{\xi}) \;\;, \end{aligned}$$

from which equation (6) of [BK1] is obtained:

$$\frac{\partial}{\partial t}\psi_n^c(\boldsymbol{\xi}) = -\dot{U}_n(\boldsymbol{\xi})\,\psi_n^c(\boldsymbol{\xi}) - \frac{1}{2}\sum_{I\subseteq\{1,\dots,n\}} \dot{U}\left(P_I\boldsymbol{\xi};P_{I^c}\boldsymbol{\xi}\right)\,\psi^c(P_I\boldsymbol{\xi})\,\psi^c(P_{I^c}\boldsymbol{\xi}) \tag{3.13}$$

satisfying initial condition $\boldsymbol{\psi}_0^c = \left(\psi_{0,n}^c(\boldsymbol{\xi})\right)_{n\geq 0}$ at t_0 with

$$\psi_0^c\left(P_I\boldsymbol{\xi}\right) = \begin{cases} 1 & \text{if } |I| = 1 \\ 0 & \text{otherwise} \end{cases} \quad . \tag{3.14}$$

For brevity, we have omitted the time dependence in (3.13). Here, $U_n(t,\boldsymbol{\xi}) = U(t,P_n\boldsymbol{\xi})$ is the energy of state $P_n\boldsymbol{\xi}$ and

$$U\left(t,P_I\boldsymbol{\xi};P_J\boldsymbol{\xi}\right) = \sum_{i\in I,\,j\in J}\sigma_i\sigma_j v\left(t,x_i-x_j\right) \quad . \tag{3.15}$$

The minus sign in the r.h.s. of (3.13) comes from the derivatives of $e^{i\phi(\xi)}$ at $\phi = 0$.

Using the variation of constants formula, (3.13) is equivalent to the following integral equation

$$\psi_n^c(t,\boldsymbol{\xi}) = -\frac{1}{2}\int_{t_0}^t ds\, e^{-\int_s^t d\tau\,\dot{U}_n(\tau,\boldsymbol{\xi})}\sum_{I\subseteq\{1,\dots,n\}}\dot{U}\left(s,P_I\boldsymbol{\xi};P_{I^c}\boldsymbol{\xi}\right)\psi^c(s,P_I\boldsymbol{\xi})\,\psi^c(s,P_{I^c}\boldsymbol{\xi})\,, \tag{3.16}$$

for $n > 1$ with $\psi_0^c(t,\boldsymbol{\xi}) = 0$ and $\psi_1^c(t,\boldsymbol{\xi}) = 1$ for all $t\in[t_0,t_1]$.

The second Ursell function ψ_2^c satisfies

$$\begin{aligned}\psi_2^c(t,\xi_1,\xi_2) &= -\int_{t_0}^t ds\,\sigma_1\sigma_2\dot{v}\left(s,x_1-x_2\right)e^{-\sigma_1\sigma_2\int_s^t d\tau\,\dot{v}\left(\tau,x_1-x_2\right)} \\ &= e^{-\sigma_1\sigma_2\int_{t_0}^t d\tau\,\dot{v}\left(\tau,x_1-x_2\right)} - 1\end{aligned} \tag{3.17}$$

and agrees with (3.3) if $\beta\vartheta(\xi_1,\xi_2) = \sigma_1\sigma_2\left(v\left(t_1,x_1-x_2\right) - v\left(t_0,x_1-x_2\right)\right)$. One can verify that there is a unique solution to the integral equation (3.16) formally given by the Ursell functions (3.3). These follows from (3.13) by induction in n. Equation (3.16), derived for the first time by Brydges–Kennedy [BK], is the starting point for the procedure of the next section.

4 Majorant Differential Equation

An one–parameter family $g(t,z)$ of holomorphic function on the open disc

$$D_R = \{z\in\mathbb{C} : |z| < R\}$$

has a convergent power series

$$g(t,z) = \sum_{j=0}^{\infty} C_j(t)\,z^j$$

which we denote by $\boldsymbol{C}\cdot\boldsymbol{z}$ with $\boldsymbol{C} = \left(C_j\right)_{j\geq 0}$ and $\boldsymbol{z} = \left(z^j\right)_{j\geq 0}$.

Definition 4.1 *$g(t,z) = \boldsymbol{C}(t)\cdot \boldsymbol{z}$ is said to be a uniform majorant of a function $f(t,z) = \boldsymbol{c}(t)\cdot\boldsymbol{z}$ in $\mathscr{D} = [a,b]\times D_R$ if*

$$|c_j(t)| \le C_j(t)$$

holds for every $j \in \mathbb{N}$ and $t \in [a,b]$. We write $f \trianglelefteq g$ for the majorant relation.

Note that a majorant g is a real analytic function and the majorant relation is preserved by differentiation in z and integration with respect to t.

Proof of Theorem 2.2. Part I. Our aim is to find an equation for a uniform majorant $\Phi(t,z) = \boldsymbol{A}(t)\cdot\boldsymbol{z}$ of the pressure (3.1). Let $\boldsymbol{a}(t) = (a_n(t))_{n\ge1}$ be given by

$$a_n(t) = \sup_{\xi_1\in\Lambda}\int d\rho(\xi_2)\cdots\int d\rho(\xi_n)\ |\psi_n^c(t,\boldsymbol{\xi})| \tag{4.1}$$

and define

$$\Gamma(t) := \|\dot v(t)\|\ , \qquad\qquad \gamma(s,t) := \int_s^t \dot B(\tau)\,d\tau\,, \tag{4.2}$$

where B and $\|\cdot\|$ are given by (2.8) and (2.6). It follows from (3.16) that

$$a_n(t) \le \frac{1}{2}\int_{t_0}^t ds\, e^{n\gamma(s,t)}\Gamma(s)\sum_{k=1}^{n-1}\binom{n}{k}k\,(n-k)\,a_k(s)\,a_{n-k}(s) \tag{4.3}$$

holds for $n>1$ with $a_1 = 1$. We thus have

$$|b_{\Lambda,n}| \le \frac{1}{n!}a_n(t_1) \le A_n(t_1) \tag{4.4}$$

uniformly in Λ where $b_{\Lambda,n}$ is the Mayer coefficients (3.2) and $\boldsymbol{A}(t) = (A_n(t))_{n\ge1}$ is a sequence of continuous functions so that $(n!A_n(t))_{n\ge1}$ satisfies (4.3) as an equality:

$$A_n(t) = \frac{1}{2}\int_{t_0}^t ds\, e^{n\gamma(s,t)}\Gamma(s)\sum_{k=1}^{n-1}k\,A_k(s)\,(n-k)\,A_{n-k}(s) \tag{4.5}$$

for $n>1$ with $A_1 = 1$.

Writing

$$\Phi\,(t,z) := \sum_{n=1}^{\infty}A_n(t)\,z^n \tag{4.6}$$

equations (3.1), (4.4) and (4.5) yields

$$\beta p_\Lambda(\beta,z) \trianglelefteq \Phi\,(t_1,z) \tag{4.7}$$

uniformly in Λ where $\Phi\,(t_1,z)$ is the solution of

$$\Phi(t,z) = z + \frac{1}{2}\int_{t_0}^t ds\,\Gamma(s)\left(z\,\frac{\partial\Phi}{\partial z}\left(s, z\,e^{\gamma(s,t)}\right)\right)^2 \tag{4.8}$$

at time $t = t_1$. Here, the following identities have been used

$$
\begin{aligned}
\sum_{n=2}^{\infty} z^n \sum_{k=1}^{n-1} kA_k \,(n-k)A_{n-k} &= \sum_{n-1=1}^{\infty} \sum_{k=1}^{n-1} kA_k z^k \,(n-k)A_{n-k} z^{n-k} \\
&= \sum_{k=1}^{\infty} kA_k z^k \sum_{n-1=k}^{\infty} (n-k)A_{n-k} z^{n-k} = \left(z\frac{\partial \Phi}{\partial z} \right)^2 .
\end{aligned}
$$

The partial derivative of Φ with respect to t and z are denoted by Φ_t and Φ_z. From (4.6) and (4.8), we have

$$
\begin{aligned}
\left(z\Phi_z\left(s, ze^{\gamma}\right)\right)_t &= \frac{\partial \gamma}{\partial t}\left(ze^{\gamma} + 2^2 z^2 e^{2\gamma} A_2(s) + 3^2 z^3 e^{3\gamma} A_3(s) + \cdots\right) \\
&= \dot{B} z e^{\gamma} \left(z\Phi_z\right)_z \left(s, ze^{\gamma}\right) ,
\end{aligned}
$$

with $\gamma = \gamma(s,t)$, and

$$
z\Phi_z = z + \int_{t_0}^{t} ds\, \Gamma(s)\, z\Phi_z\left(s, ze^{\gamma(s,t)}\right) \; ze^{\gamma(s,t)} \left(z\Phi_z\right)_z \left(s, ze^{\gamma(s,t)}\right) .
$$

Differentiating (4.8) with respect to t,

$$
\Phi_t = \frac{1}{2}\Gamma (z\Phi_z)^2 + \int_{t_0}^{t} ds\, \Gamma(s) \;\; z\Phi_z\left(s, ze^{\gamma(s,t)}\right) \;\; \left(z\Phi_z\left(s, ze^{\gamma(s,t)}\right)\right)_t
$$

thus yields

$$
\Phi_t = \frac{1}{2}\Gamma (z\Phi_z)^2 + \dot{B}\left(z\Phi_z - z\right)
$$

with initial condition $\Phi(t_0, z) = z$. This establishes the majorant relation of Theorem 2.2 and equation (2.19).

□

It will be convenient to consider the density

$$
\rho_\Lambda(\beta, z) = z\frac{\partial p_\Lambda}{\partial z}(\beta, z) \tag{4.9}
$$

whose corresponding Mayer series has the same radius of convergence of the Mayer series of p_Λ. Writing $\Theta(t,z) = \Phi_z(t,z)$, we have

$$
\frac{\beta}{z}\rho_\Lambda(\beta, z) \trianglelefteq \Theta(\beta, z) \tag{4.10}
$$

with Θ satisfying the following quasi–linear first order PDE

$$
\Theta_t = \Gamma\left(z\Theta^2 + z^2\Theta\Theta_z\right) + \dot{B}\left(z\Theta_z + \Theta - 1\right) \tag{4.11}
$$

with initial condition $\Theta(t_0, z) = 1$.

In view of (4.6), (4.9) and (4.10), we seek for solutions of (4.11) of the form

$$\Theta(t,z) = 1 + \sum_{n=2}^{\infty} C_n(t) z^{n-1} \tag{4.12}$$

with $C_n = nA_n(t) \geq 0$ and this form is preserved by the equation.

Remark 4.2 *The convergence of Mayer series for an integrable stable potential ($v(t)$ so that $\int_{t_0}^{t_1} \|\dot{v}(t)\|\, dt < \infty$ and $v(t_1) = \beta\vartheta$) can be established as in the case of non negative potential ($\dot{B} \equiv 0$). Writing $w = e^{\gamma(t_0,t)} z$ and $\widetilde{A}_n = e^{-n\gamma(t_0,t)} A_n$, we have*

$$\Phi(t,z) = \sum_{n=1}^{\infty} \widetilde{A}_n(t)\ w^n := \widetilde{\Phi}(t,w)$$

with $\widetilde{\Phi}$ satisfying $\widetilde{\Phi}_t = (\Gamma/2)\left(w\widetilde{\Phi}_w\right)^2$. The Mayer series of these systems converges for all (β, z) so that the classical solution $\widetilde{\Phi}(t, e^{\gamma(t_0,t_1)}|z|)$ exists up to $t = t_1$.

5 The Method of Characteristics

In this section the convergence of Mayer series will be addressed from the point of view of (4.11). We seek for a subset of $\{(t,z) : t \in [t_0, t_1], z \geq 0\}$ in which there exists a unique classical solution of the initial value problem. We shall distinguish two cases, depending on whether the stability function $\dot{B}(t)$ is or is not identically zero.

5.1 Non Negative Potentials

Let equation (4.11) with $\dot{B}(s) \equiv 0$,

$$\Theta_t = \Gamma\left(z\Theta^2 + z^2\Theta\Theta_z\right) \tag{5.1}$$

with $\Theta(t_0, z) = 1$, be addressed by the method of characteristics. This equation describes, by (4.10), a density majorant of a gas with non negative (repulsive) potential $v \geq 0$.

It is convenient change the t variable by $\tau = \int_{t_0}^{t} \Gamma(s)\ ds$ and write $U(\tau) = \Theta(t(\tau), z(\tau))$. Equation

$$U' = t'\Theta_t + z'\Theta_z$$

together with (5.1) yield a pair of ordinary differential equations (ODE)

$$U' = zU^2 \tag{5.2}$$

$$z' = -z^2 U \tag{5.3}$$

satisfying the initial conditions $U(0) = 1$ and $z(0) = z_0$. Dividing (5.2) by (5.3) gives

$$\frac{dU}{dz} = -\frac{U}{z}$$

which can be solved as a function of z

$$U(z) = \frac{z_0}{z} \ . \tag{5.4}$$

Inserting (5.4) into (5.3), leads to the following characteristic equation $z' = -z_0 z$ whose solution is

$$z(\tau) = z_0 \exp\{-z_0\tau\} \ . \tag{5.5}$$

Now, $\Theta(t, z)$ is given by (5.4) with $z_0 = z_0(\tau, z)$ implicitly given by (5.5). If $W(x)$ denotes the Lambert W–function [K], defined as the inverse of $f(W) = We^W$ ($f(W(x)) = x$ and $W(f(w)) = w$), we have

$$z_0(\tau, z) = \frac{-1}{\tau} W(-z\tau)$$

and the solution of the initial value problem (5.1) is thus given by

$$\Theta(t, z) = \frac{-1}{z\tau} W(-z\tau)$$

provided

$$ez\tau = ez \int_{t_0}^{t} \Gamma(s)\, ds < 1 \ . \tag{5.6}$$

Since $\Im(W(x)) \neq 0$ for $x < -1/e$ in the principal branch of the W –function (see e.g. [K]), this condition has to be imposed to ensure uniqueness. W has a branching point singularity at $-1/e + i0$ with $W(-1/e) = \Re(W(-1/e)) = -1$. At the singularity $z^* = z^*(t) = (z\tau)^{-1}$, $\Theta(t, z^*) = e$ and the series (4.12) diverges.

Remark 5.1 *1. For non negative potentials we can actually find a pressure majorant $\Phi(\beta, z)$ explicitly. Differentiating $x = W(x)\exp(W(x))$ and solving for W' gives, after some manipulations,*

$$\frac{W}{x} = \left(W + \frac{W^2}{2}\right)'$$

and this, together with (4.9) and $W(0) = 0$, yields

$$\Phi(t, z) = \frac{-1}{\tau} \int_0^{-z\tau} \frac{W(x)}{x} dx = \frac{-1}{\tau}\left(W(-z\tau) + \frac{1}{2} W^2(-z\tau)\right) \ .$$

2. The interpretation of (5.6) is as follows. Let $\mathscr{T}(r, z_0) = (\ln r - \ln z_0)/(r - z_0)$ be the time necessary for two characteristics starting from r and z_0 to meet at a point z:

$$z = re^{-r\mathscr{T}} = z_0 e^{-z_0\mathscr{T}} \ .$$

$\mathscr{T}(r, z_0)$ is a monotone decreasing function of r with $\lim_{r\nearrow z_0} \mathscr{T}(r, z_0) = 1/z_0$. Note that, $\Theta(\mathscr{T}, z)$ assumes two different values $U(z) = r/z$ and $U_0(z) = z_0/z$ for $r < z_0$ which tends to e in the limit $r \nearrow z_0$. The minimum intercepting time of two characteristics with $r \leq z_0$ is attained when both coincide and this occurs exactly at the branching singularity $-z\mathscr{T} = -1/e$ of $W(x)$ where uniqueness of (5.1) breaks down.

3. From condition (5.6) and (4.7), the Mayer series (3.1) converges if

$$|z| < \left(e \int_{t_0}^{t_1} \Gamma(s)\ ds \right)^{-1} . \tag{5.7}$$

Choosing $v(t) = (t - t_0)\beta\vartheta/(t_1 - t_0)$, *we have* $\int_{t_0}^{t_1} \Gamma(s)\ ds = \beta \|\vartheta\|$ *and for stable potentials, according to Remark 4.2, the Mayer series converges for all* (β, z) *so that*

$$e\,|z| \exp\left\{ \int_{t_0}^{t_1} \dot{B}(s)\ ds \right\} \int_{t_0}^{t_1} \Gamma(s)\ ds\ < 1 \tag{5.8}$$

which gives the same condition $e\beta\,|z|\,e^{\beta B}\,\|\vartheta\| < 1$ *obtained by Brydges and Federbush (see eq. (11) in [BF]).*

5.2 Stable Potentials

We now turn to equation (4.11). Instead of solving it explicitly, we shall obtain an upper solution $\overline{\Theta}(t, z)$ whose existence implies convergence of density majorant series. $\overline{\Theta}(t, z)$ satisfies equation (5.1) for non–negative potential with Γ substituted by Γ/f for some properly chosen functions f. Our procedure allows us to obtain various convergence conditions depending on the choice of f.

Proof of Theorem 2.2. Conclusion. Let us change the variable z by $w = f(t)z$, for a function $f(t) \geq 1$ such that $f(0) = 1$, and define $\Theta(t, z) = f(t)\Psi\left(t, f(t)z\right)$. Using the chain rule, we have

$$\begin{aligned} \Theta_t &= f\left(\Psi_t + \frac{f'}{f}\Psi + \frac{f'}{f} w\Psi_w \right) \\ z\Theta_z &= f w \Psi_w \end{aligned}$$

which, together with (4.11), yields

$$\Psi_t - \left(\dot{B} - \frac{f'}{f} + \Gamma w \Psi \right) w\Psi_w = \left(\dot{B} - \frac{f'}{f} \right) \Psi + \Gamma w \Psi^2 - \frac{\dot{B}}{f} .$$

Choosing

$$f(t) = \exp\left\{ \int_{t_0}^{t} \dot{B}(s)\ ds \right\}$$

yields

$$\Psi_t - \Gamma w^2 \Psi \Psi_w = \Gamma w \Psi^2 - \frac{\dot{B}}{f} \leq \Gamma w \Psi^2 \tag{5.9}$$

by positivity of f and $\dot{B}$. Note that Ψ satisfies (5.1) as an inequality.

As before, we define $\tau = \int_{t_0}^{t} \Gamma(s)\ ds$ and write

$$U(\tau) = \Psi(t(\tau), w(\tau)). \tag{5.10}$$

From $U' = \Psi_t t' + \Psi_w w'$and (5.9), we have

$$U' \leq wU^2 \tag{5.11}$$

$$w' = -w^2 U \tag{5.12}$$

with initial conditions $U(0) = 1$ and $w(0) = w_0$. Equations (5.11) and (5.12) can be written in terms of the variables $V(\tau) = wU$ and w as

$$V' \leq 0$$

$$\frac{w'}{w} = -V \tag{5.13}$$

with $V(0) = w_0$, which implies

$$V(\tau) \leq w_0 \tag{5.14}$$

and

$$w(\tau) \geq w_0 \exp\{-w_0 \tau\} \ . \tag{5.15}$$

Here, we need an inequality for $w_0 = w_0(\tau, z)$ so that (5.15) holds. Writing (5.15) as

$$-w_0 \tau \ e^{-w_0 \tau} \geq -w\tau \ ,$$

we have

$$w_0(\tau, w) \leq -\frac{1}{\tau} W(-w\tau) \ , \tag{5.16}$$

where W is the Lambert W–function, provided

$$w\tau \ = z \exp\left\{\int_{t_0}^{t} \dot{B}(s') \ ds'\right\} \int_{t_0}^{t} \Gamma(s) \ ds \ < 1/e \ .$$

Plugging (5.16) into (5.14), together with (5.10), leads to

$$\Theta(t, z) = \frac{V(\tau(t))}{z} \leq \overline{\Theta}(t, z) \ , \tag{5.17}$$

where

$$\overline{\Theta}(t, z) = \frac{-1}{z \displaystyle\int_{t_0}^{t} \Gamma(s) \ ds} W\left(-z \exp\left\{\int_{t_0}^{t} \dot{B}(s) \ ds\right\} \int_{t_o}^{t} \Gamma(s) \ ds \right) \ ,$$

recovering the statements of Remark 4.2.

We now go beyond inequality (5.15). For this we modify slightly the previous scaling by defining $\Theta(t, z) = \Psi^{(1)}(t, f_1(t) z)$. By the chain rule, $\Psi^{(1)}$ is the solution of

$$\Psi_t - \frac{\Gamma}{f_1} w^2 \Psi \Psi_w = \left(\dot{B} - \frac{f_1'}{f_1}\right) w \Psi_w + \dot{B}\Psi - \dot{B} + \frac{\Gamma}{f_1} w \Psi^2$$

satisfying $\Psi^{(1)}(t_0, w) = 1$. Choosing

$$f_1(t) = \exp\left\{2 \int_{t_0}^{t} \dot{B}(s) \ ds\right\} \tag{5.18}$$

and using

$$-w\Psi_w^{(1)} + \Psi^{(1)} - 1 \;=\; -\sum_{n=2}^{\infty}(n-1)C_n^{(1)}w^{n-1} + \sum_{n=1}^{\infty}C_n^{(1)}w^{n-1} - 1$$

$$= \; -\sum_{n=2}^{\infty}(n-2)C_n^{(1)}w^{n-1} \leq 0$$

in view of (4.12) and $w = zf_1(t) \geq 0$, we have

$$\Psi_t - \frac{\Gamma}{f_1}w^2\Psi\Psi_w = \dot{B}\left(-w\Psi_w + \Psi - 1\right) + \frac{\Gamma}{f_1}w\Psi^2 \leq \frac{\Gamma}{f_1}w\Psi^2 \ . \tag{5.19}$$

We define $\tau = \int_{t_0}^{t}\Gamma(s)/f_1(s)\ ds$ and repeat the steps from (5.10) to (5.17). Provided

$$w\tau \ = z\tau_1 < 1/e \tag{5.20}$$

holds with τ_1 given by

$$\tau_1(t) \ = \int_{t_0}^{t} e^{2\int_s^t \dot{B}(s)\ ds}\,\Gamma(s)\ ds \ , \tag{5.21}$$

the solution of the initial value problem (4.11) satisfies

$$\Theta(t,z) \leq \ \frac{-1}{\tau_1 z}W\left(-\tau_1 z\right) \ . \tag{5.22}$$

It follows from (5.17), (5.22) and Remark 5.1.1. that (2.20) holds and converges either under the condition (5.8) or under (5.20). This concludes the proof of Theorem 2.2.

□

Remark 5.2 *1. Working directly with the integral equation (3.16), Brydges–Kennedy [BK1] have established recursively*

$$a_n(t) \leq {\tau_1}^{n-1}n^{n-2}$$

with τ_1 and a_n given by (5.21) and (4.3). Hence, the density majorant of stable potentials satisfies

$$\Theta(t,z) \;=\; \sum_{n=1}^{\infty}\frac{a_n(t)}{(n-1)!}z^{n-1}$$

$$\leq \; \frac{-1}{\tau_1 z}\sum_{n=1}^{\infty}\frac{(-n)^{n-1}}{n!}\left(-\tau_1 z\right)^n \ ,$$

and this, by the Taylor series of the Lambert W–function [K], is exactly (5.22).

2. *Criterion (5.20) for convergence may be more advantageous than (5.8) when $v(t)$ is a multi-scale decomposition of a potential. As observed by Brydges–Kennedy (see Remarks at the end of [BK1]), a poor stability bound (large $\dot{B}(t)$) at short distances ($t \ll 0$) may be overcome by the smallness of the interaction $\Gamma = \|\dot{v}(t)\|$ at these scales.*

6 Application to Yukawa Gas

The majorant method is applied to investigate whether the Mayer series for the Yukawa gas is convergent in the region of collapse. The first and third subsections redo the proofs in [B, BK] and [BK] for $\beta < 4\pi$ and $4\pi \leq \beta < 16\pi/3$, respectively. After introducing the model we illustrate how useful is condition (5.20) for $\beta < 4\pi$. The procedure of Subsection 5.2 is then modified in order to overcome the first threshold at $\beta_1 = 4\pi$. In the present section we intend to show the capabilities of our majorant method to extend results in [BK]. The proof of Theorem 2.3 will be deferred to Section 7.

6.1 Convergence of the Mayer Series for $\beta < 4\pi$

Particles in this system have assigned two charges $\sigma \in \{-1, 1\}$ and they interact via the Yukawa potential

$$\begin{aligned} \vartheta(\xi_1, \xi_2) &= \sigma_1\sigma_2 \left(-\Delta + 1\right)^{-1} (x_1 - x_2) \\ &= \frac{\sigma_1\sigma_2}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{e^{-ik\cdot(x_1-x_2)}}{k^2+1} d^2k \end{aligned} \tag{6.1}$$

where $-\Delta$ is the Laplace operator in $\mathbb{R}^2$. Potential (6.1) violates condition 1. of (2.5), since

$$|\vartheta(\xi, \xi)| = \frac{1}{4\pi} \lim_{R\to\infty} \int_0^R \frac{2\eta}{\eta^2+1} d\eta = \frac{1}{4\pi} \lim_{R\to\infty} \ln\left(R^2+1\right)$$

diverges logarithmically. We replace $\beta\vartheta(\xi_1, \xi_2)$ by $\sigma_1\sigma_2 v(t, x_1 - x_2)$ given by

$$v(t, x) = \beta \left(-\Delta + \kappa(t)\right)^{-1} (x) = \frac{\beta}{2\pi} \, K_0\left(\sqrt{\kappa(t)}\,|x|\right) \tag{6.2}$$

where K_0 is the modified Bessel function of second kind (see e.g. [GJ] p. 126) and κ is a monotone decreasing scale function such that

$$\kappa(-\infty) = \infty \qquad \text{and} \qquad \kappa(0) = 1 \ ,$$

and decompose it into scales

$$v(t, x) = \int_{-\infty}^{t} \dot{v}(s, x) \; ds$$

where

$$\dot{v}(s, x) = \frac{\beta}{4\pi} \frac{\kappa'(s)}{\sqrt{\kappa(s)}} \, |x| \, K_0'\left(\sqrt{\kappa(s)}\,|x|\right) \ . \tag{6.3}$$

By definitions (2.8) and (4.2), we have

Proposition 6.1 *For each $-\infty < s < 0$, $\dot{v}(s, x - y)$ is a positive kernel satisfying properties $1-3$ of (2.5) with*

$$\dot{B}(s) = \frac{1}{2}\,|\dot{v}(s, 0)| = \frac{\beta}{8\pi}\,(-\ln\ \kappa(s))' \tag{6.4}$$

and

$$\Gamma(s) = \|\dot{v}(s, \cdot)\| = \beta\,\frac{-\kappa'(s)}{\kappa^2(s)}. \tag{6.5}$$

Proof. Properties $1-3$ of (2.5) follow immediately from the expressions (6.2) and (6.1), provided (6.4) and (6.5) are established. Applying Cauchy formula for the integral (6.1) (with 1 replaced by $\kappa(t)$) in the $x-y$ direction, gives

$$v(t,x-y)=\frac{\beta}{2\pi}g\left(\sqrt{\kappa(t)}\,|x-y|\right)$$

with

$$g(w):=\int_0^\infty \frac{e^{-w\sqrt{k^2+1}}}{\sqrt{k^2+1}}dk\ . \tag{6.6}$$

By the chain rule

$$(g\circ w)'=-w'\int_0^\infty e^{-w\sqrt{k^2+1}}dk\ =-\frac{w'}{w}\int_w^\infty \frac{e^{-\zeta}}{\sqrt{\zeta^2-w^2}}\zeta d\zeta\equiv-\frac{w'}{w}h(w) \tag{6.7}$$

and equation (6.3) can thus be written as

$$\dot v(s,x-y)=\frac{-\beta}{4\pi}\frac{\kappa'(s)}{\kappa(s)}h\left(\sqrt{\kappa(s)}\ \ |x-y|\right)\ , \tag{6.8}$$

which gives (6.4) by taking $|x-y|=0$. Note that $-\kappa'(s)\geq 0$, by assumption. Integrating (6.6) times w, yields

$$\int_0^\infty g(w)wdw=\int_0^\infty \frac{1}{(k^2+1)^{3/2}}dk=1 \tag{6.9}$$

which implies

$$\begin{aligned}\|\dot v(s,\cdot)\| &= \frac{\beta}{2\pi}\frac{d}{ds}\int_0^\infty g\left(\sqrt{\kappa(s)}r\right)2\pi r dr\\ &= \beta\left(\frac{1}{\kappa(s)}\right)'\end{aligned}$$

and concludes the proof.

□

The majorant density $\Theta(t,z)$ of the Yukawa gas satisfies equation (4.11) with $\dot B$ and Γ given by (6.4) and (6.5), respectively, and initial data $\Theta(t_0,z)=1$ where t_0 plays the role of a short distance cutoff, which is removed if $t_0\to-\infty$.

By (4.10) and Remark 5.2.2, the Mayer series converges if (5.20) holds. We have

$$\lim_{t_0\to-\infty}\tau_1(0)=-\beta\lim_{t_0\to-\infty}\int_{t_0}^0\frac{\kappa'(s)}{\kappa^{2-\beta/4\pi}(s)}ds=\frac{4\pi\beta}{4\pi-\beta}\lim_{t_0\to-\infty}\left\{1-\kappa^{\beta/4\pi-1}(t_0)\right\}=\frac{4\pi\beta}{4\pi-\beta}$$

provided $\beta<4\pi$, independently of the scale function κ. The Mayer series of Yukawa gas converges if

$$\beta<4\pi\qquad\text{and}\qquad |z|<\frac{4\pi-\beta}{4\pi e\beta}\ .$$

Although it has been previously established in [B, BK], our method is independent of the scale function κ and improves their radius of convergence.

6.2 The 2–Regularized Majorant Solution

The Mayer series for the Yukawa gas between the first and second thresholds, $4\pi \leq \beta < 6\pi$, diverges because the norm $a_2(t)$ of second Ursell function $\psi_2^c(t, \boldsymbol{\xi})$ diverges when the ultraviolet cutoff is removed (see e.g. [BGN, BK] and references therein). We are going to show that the regularized density majorant $\Theta^{(2)}(t,z)$, defined by $\Theta(t,z)$ with the singular part of $a_2(t)$ removed by a Lagrange multiplier, exists and implies convergence of the Mayer series for $\beta < 16\pi/3$. For $16\pi/3 \leq \beta < 6\pi$, the upper bound A_3 for the norm of $\psi_3^c(t,\boldsymbol{\xi})$ also diverges and equation (4.3) with $n = 3$ has to be modified in order the majorant relation (4.10) holds with Θ replaced by $\Theta^{(2)}$. The modified equation requires an improved stability condition which prevents $\beta_2 = 16\pi/3$, and more generally $\beta = \beta_{2r}$, $r = 1, 2, \ldots$, where

$$\beta_n = 8\pi \left(1 + \frac{1}{n}\right)^{-1} = 8\pi \frac{n}{n+1} , \tag{6.10}$$

to be considered a threshold of the Yukawa model (see Remarks on pg. 47 of [BK]).

The second Ursell function $\psi_2^c(t,\boldsymbol{\xi})$ is explicitly given by (3.17). Together with (6.2), we have

$$\psi_2^c(t,\xi_1,\xi_2) = \exp\left\{\frac{-\beta\sigma_1\sigma_2}{2\pi} \left(K_0\left(\sqrt{\kappa(t)}r\right) - K_0\left(\sqrt{\kappa(t_0)}r\right)\right)\right\} - 1 \tag{6.11}$$

where $r = |x_1 - x_2|$.

To isolate the singularity of (6.11) we replace $a_2(t) = \sup_{\xi_1\in\Lambda} \displaystyle\int d\rho(\xi_2)\, |\psi_2^c(t,\xi_1,\xi_2)|$ by

$$a_2^{\pm}(t) = \sup_{x_1} \int d^2x_2 \frac{1}{2} \sum_{\substack{\sigma_1,\sigma_2\in\{-1,1\}:\\ \sigma_1\sigma_2=\pm}} |\psi_2^c(t,\xi_1,\xi_2)|$$

which distinguish whether the charge product $\sigma_1\sigma_2$ is positive or negative ($\sigma_1\sigma_2 = -1 \Leftrightarrow \sigma_1 + \sigma_2 = 0$). Using the fact that $K_0(x)$ is a monotone decreasing function together with $1 - e^{-y} \leq y$ for $y \geq 0$, we have

$$\begin{aligned} a_2^{+}(t) &\leq \beta \int_0^\infty r \left(K_0\left(\sqrt{\kappa(t)}r\right) dr - K_0\left(\sqrt{\kappa(t_0)}r\right)\right) dr \\ &\leq \beta \left(\frac{1}{\kappa(t)} - \frac{1}{\kappa(t_0)}\right) . \end{aligned}$$

For $\sigma_1\sigma_2 = -1$, the first line of (3.17), $\dot v(s,r) \geq 0$ and (6.8), imply

$$\begin{aligned} a_2^{-}(t) &= \frac{\beta}{2} \int_0^\infty dw\; w\; h(w) \int_{t_0}^t \frac{-\kappa'(s)}{\kappa^2(s)} \exp\left\{\frac{\beta}{4\pi} \int_s^t \frac{-\kappa'(\tau)}{\kappa(\tau)} h\left(\sqrt{\kappa(\tau)/\kappa(s)}w\right) d\tau\right\} ds \qquad (6.12) \\ &\leq \frac{2\pi\beta}{4\pi - \beta} \int_0^\infty dw\; w\; h(w) \left(\frac{1}{\kappa(t)} - \frac{1}{\kappa(t_0)} \left(\frac{\kappa(t_0)}{\kappa(t)}\right)^{\beta/4\pi}\right) \\ &= \frac{4\pi\beta}{4\pi - \beta} \left(\frac{1}{\kappa(t)} - \frac{1}{\kappa(t_0)} \left(\frac{\kappa(t_0)}{\kappa(t)}\right)^{\beta/4\pi}\right) . \end{aligned}$$

Note that $h(w)$ is a monotone decreasing function with $h(0) = 1$ and

$$\int_0^\infty dw\ w\ h(w) = \int_0^\infty dw\ w^2\ g'(w) = 2\int_0^\infty dw\ w\ g(w) = 2$$

by (6.7), integration by parts and (6.9). As a consequence, both a_2^+ and a_2^- converges as $t_0 \to \infty$ if $\beta < 4\pi$ (a_2^+ remains bounded for any $\beta > 0$). If $\beta > 4\pi$, since the main contribution of (6.12) comes from a neighborhood of $w = 0$, there exist $\varepsilon > 0$ and a constant $C = C(\varepsilon) > 0$ such that $\dfrac{\beta}{4\pi} - \varepsilon > 1$ and

$$a_2^-(t) \geq C \left(\frac{1}{\kappa(t_0)} \left(\frac{\kappa(t_0)}{\kappa(t)} \right)^{\beta/4\pi - \varepsilon} - \frac{1}{\kappa(t)} \right) \tag{6.13}$$

diverges as $t_0 \to -\infty$.

Now we shall write the majorant equation (4.11) with the singularity of A_2 removed by a Lagrange multiplier. This can be done in several ways. We may define Θ^R such that $\Theta_z^R(t,0) = 0$ for all $t \geq 0$, as the solution of

$$\Theta_t = \Gamma\left(z\Theta^2 + z^2\Theta\Theta_z\right) + \dot{B}\left(z\Theta_z + \Theta - 1\right) - z\left(\Gamma + 2\dot{B}\Theta_z(t,0)\right)$$

with $\Theta(0,z) = 1$. Note that $d\Theta_z^R(t,0)/dt = \Theta_{tz}^R(t,0) = 0$ holds for all $t \geq 0$. Another way adopt here is as follows. The linear part of equation (4.11) is responsible for the singularity of A_2 if $\beta > 4\pi$ and we may choose, instead, a Lagrange multiplier proportional to $\dot{B}$

$$\Theta_t = \Gamma\left(z\Theta^2 + z^2\Theta\Theta_z\right) + \dot{B}\left(z\Theta_z + \Theta - 1\right) - \frac{1}{2}z\dot{B}\Theta_z(t,0) \tag{6.14}$$

(with a quarter of intensity needed to make A_2 vanishes).

Let us denote the solution of (6.14) by $\Theta^{(2)}$. As in Subsection 5.2, if $\Theta^{(2)}(t,z) = \Psi^{(2)}(t,w)$, where $w = f_2(t)z$ and

$$f_2(t) = \exp\left\{\frac{3}{2}\int_{t_0}^t \dot{B}(s)\ ds\right\}\ , \tag{6.15}$$

using (with $\Psi^{(2)} = 1 + \sum_{n=2}^\infty C_n^{(2)} w^{n-1}$, $C_n^{(2)} \geq 0$)

$$\begin{aligned} -\frac{1}{2}w\Psi_w^{(2)} + \Psi^{(2)} - 1 - \frac{1}{2}w\left(\Psi_w^{(2)}(t,0)\right) &= -\frac{1}{2}\sum_{n=2}^\infty (n-1)C_n^{(2)}w^{n-1} + \sum_{n=1}^\infty C_n^{(2)}w^{n-1} - 1 - \frac{1}{2}C_2 w \\ &= -\frac{1}{2}\sum_{n=3}^\infty (n-3)C_n^{(2)}w^{n-1} \leq 0 \end{aligned}$$

it follows that $\Psi^{(2)}$ is the solution of

$$\begin{aligned} \Psi_t - \frac{\Gamma}{f_2}w^2\Psi\Psi_w &= \left(\dot{B} - \frac{f_2'}{f_2}\right)w\Psi_w + \dot{B}\left(\Psi - 1 - \frac{1}{2}C_2 w\right) + \frac{\Gamma}{f_2}w\Psi^2 \\ &= \dot{B}\left(-\frac{1}{2}w\Psi_w + \Psi - 1 - \frac{1}{2}C_2 w\right) + \frac{\Gamma}{f_2}w\Psi^2 \\ &\leq \frac{\Gamma}{f_2}w\Psi^2 \end{aligned}$$

satisfying $\Psi^{(2)}(t_0,w) = 1$. Repeating the steps from (5.10) to (5.17) with $\tau = \int_{t_0}^{t} (\Gamma/f_2)(s)\ ds$, provided $w\tau = z\tau_2 < 1/e$ where

$$\tau_2\ (t)\ = \int_{t_0}^{t} \exp\left\{\frac{3}{2}\int_{s}^{t} \dot{B}(s')\ ds'\right\}\ \Gamma(s)\ ds, \tag{6.16}$$

the solution $\Theta^{(2)}$ of the initial value problem (6.14) satisfies

$$\Theta^{(2)}(t,z) \leq \frac{-1}{\tau_2 z} W\left(-\tau_2 z\right)\ . \tag{6.17}$$

Substituting (6.4) and (6.5) into (6.16) with $t = 0$, we have

$$\lim_{t_0 \to -\infty} \tau_2\ (0) = \beta \lim_{t_0 \to -\infty} \int_{t_0}^{0} \frac{-\kappa'(s)}{\kappa^{2-3\beta/16\pi}(s)} ds = \frac{16\pi\beta}{16\pi - 3\beta}$$

independently of the scale function κ, and the power series

$$\Theta^{(2)}(t,z) = 1 + \sum_{n=2}^{\infty} C_n(t)\ z^{n-1}$$

converges if $\beta < 16\pi/3$ and $|z| < (e\tau_2)^{-1}$.

6.3 Convergence of the Mayer Series for $4\pi \leq \beta < 16\pi/3$

$\Theta^{(2)}(0,z)$ is a candidate to be a majorant of the density $\beta\rho(\beta,z)/z$ with the $O\left(z\right)$ term omitted (see equation (4.10)). Its linear coefficient C_2, which satisfies

$$\dot{C}_2 \leq \frac{3}{2}\dot{B}C_2 + \Gamma\ , \qquad\qquad t_0 < t \leq 0 \tag{6.18}$$

with $C_2(t_0) = 0$, is positive and bounded by $16\pi\beta/(16\pi - 3\beta)$, in view of the fact τ_2, defined by (6.16), solves (6.18) as an equality. Hence, by (3.1) and (4.9), the Mayer series of Yukawa gas with the leading term removed converges uniformly in t_0 for $4\pi \leq \beta < 16\pi/3$ if

$$nb_n = n \lim_{\Lambda \nearrow \mathbb{R}^2} b_{n,\Lambda} \leq C_n \tag{6.19}$$

is shown to hold for all $n > 2$. The inequality (6.19), however, does not follow from (4.3) because the r.h.s. of that equation depends on $a_2(t) = a_2^+(t)/2 + a_2^-(t)/2$ which, as shown in Subsection 6.2, diverges as $t_0 \to \infty$. We are going to show that $a_2(t)$ in the r.h.s. of (4.3) can indeed be replaced by $a_2^+(t)$ and this implies (6.19).

Define

$$F(t, P_n\boldsymbol{\xi}) = e^{-\int_{s}^{t} d\tau\, \dot{U}_n(\tau,\boldsymbol{\xi})}\,\dot{U}\left(s, P_I\boldsymbol{\xi}; P_{I^c}\boldsymbol{\xi}\right)\psi^c(s, P_I\boldsymbol{\xi})\ , \tag{6.20}$$

and suppose $n \geq 3$ and $|I| = 2$. Using the stability bound (2.8) together with (4.1) and (4.2), we have

$$\begin{aligned}
\prod_{l\in I}\int d\rho(\xi_l)\,|F(t,P_n\boldsymbol{\xi})| &\leq e^{n\gamma(s,t)}\sum_{i\in I^c}|\sigma_i|\prod_{l\in I}\int d\rho(\xi_l)\left|\sum_{j\in I}\dot{v}(s,x_i-x_j)\sigma_j\psi^c(s,P_I\boldsymbol{\xi})\right| \\
&\leq e^{n\gamma(s,t)}\Gamma(s)\,|I^c|\sup_x\int d^2x'\frac{1}{4}\sum_{\sigma,\sigma'\in\{-1,1\}}|\sigma+\sigma'|\,|\psi_2^c(s,\xi,\xi')| \\
&= e^{n\gamma(s,t)}\Gamma(s)\,|I^c|\;\, a_2^+(s) \qquad (6.21)
\end{aligned}$$

by translational invariance. Since a_2^+ satisfies the equation

$$\dot{a}_2^+ \leq \frac{\Gamma}{2}\,, \qquad\qquad t_0 < t \leq 0\,,$$

with $a_2^+(t_0) = 0$ (for $n = 2$ the stability condition (2.21) holds with $\delta_2 = 2$), $a_2^+(t) < A_2(t) = C_2(t)/2$ holds for all $t \in [t_0, 0]$. Hence, including the modification (6.21), equation (4.5) majorates (4.3) and (6.19) holds in view of (4.4).

The Mayer series of Yukawa gas, with the singular part of the norm of ψ_2^c removed, converges if

$$\beta < \frac{16}{3}\pi \qquad \text{and} \qquad |z| < \frac{16\pi - 3\beta}{16\pi\beta e}$$

and this establishes the conjecture up to β_2.

□

Remark 6.2 *The authors of [BK] have established convergence of* Θ *for* $\beta < 16\pi/3$ *with* $O(z)$ *term omitted directly from the integral equation (3.16). Our method employs the same ingredients but the invariance under translation is used right in the beginning, and not after various manipulations of equation (6.21). This little detail improves the radius of convergence (see Theorem* 4.3 *of [BK]) and allows us to go beyond the second threshold.*

7 Proof of BGN's Conjecture

The procedure of Section 6 will now be extended in order to prove Theorem 2.3. Our proof generalizes and includes all results obtained previously.

The regularized density majorant $\Theta^{(k)}(\beta, z)$, defined by $\Theta(\beta, z)$ with the singular part removed by a Lagrange multipliers, is introduced as follows. If $\beta < \beta_k = 8\pi\left(1 + k^{-1}\right)^{-1}$ the k–regularized majorant is the solution of equation

$$\Theta_t = \Gamma\left(z\Theta^2 + z^2\Theta\Theta_z\right) + \dot{B}\left(z\Theta_z + \Theta - L_k\right) \qquad (7.22)$$

where

$$L_k = L_k(t) = 1 + \sum_{j=1}^{k-1}\frac{1}{j!}\left(1 - \frac{j}{k}\right)z^j\left(\Theta_{\underbrace{z\cdots z}_{j-\text{times}}}(t,0)\right)$$

with $\Theta(0,z)=1$. We analogously define $\Theta^{(k)}(t,z)=\Psi^{(k)}(t,w)$, where $w=f_k(t)z$ and

$$f_k(t)=\exp\left\{\frac{k+1}{k}\int_{t_0}^{t}\dot{B}(s)\;ds\right\}\;. \tag{7.23}$$

Using

$$\left(\dot{B}-\frac{f_k'}{f_k}\right)w\Psi_w^{(k)}+\dot{B}\left(\Psi^{(k)}-1\right)-\dot{B}\sum_{m=2}^{k}\left(1-\frac{m-1}{k}\right)C_m^{(k)}w^{m-1}$$

$$=-\frac{1}{k}\sum_{m=k+1}^{\infty}(m-k-1)C_m^{(k)}w^{m-1}\le 0$$

together with (7.22), $\Psi^{(k)}$ is the solution of

$$\Psi_t-\frac{\Gamma}{f_k}w^2\Psi\Psi_w\le\frac{\Gamma}{f_k}w\Psi^2$$

satisfying $\Psi^{(k)}(0,w)=1$. Repeating the steps from (5.10) to (5.17), the solution $\Theta^{(k)}$ of the initial value problem (7.22) satisfies

$$\Theta^{(k)}(t,z)\le\frac{-1}{\tau_k z}W\left(-\tau_k z\right) \tag{7.24}$$

provided $\tau_k z<1/e$ where, by (6.10),

$$\tau_k=\int_{t_0}^{t}\exp\left\{\frac{k+1}{k}\int_{s}^{t}\dot{B}(s')\;ds'\right\}\;\Gamma(s)\;ds\;=\frac{\beta\beta_k}{\beta_k-\beta}\left(\frac{1}{\kappa(t)}-\frac{1}{\kappa(t_0)}\left(\frac{\kappa(t_0)}{\kappa(t)}\right)^{\beta/\beta_k}\right) \tag{7.25}$$

Hence, (7.24) defines a convergent power series in z , uniformly in t_0, if $\beta<\beta_k$ and $|z|<(e\tau_k)^{-1}$.

We now examine whether $\Theta^{(2r+1)}(0,z)$ majorizes for $\beta<\beta_{2r+1}$ the density

$$\frac{\beta}{z}\rho^{(r)}(\beta,z)=\frac{\beta}{z}\rho_\Lambda-\left(2b_{\Lambda,2}z+\cdots+2rb_{\Lambda,2r}z^{2r-1}\right)\;,$$

with the leading odd terms up to order $2r-1$ omitted. Because the r.h.s. of equation (4.3) depends on $a_{2j}(t)$, $j=1,\ldots,r$, which diverges as well as equation (6.13) as $t_0\to\infty$, inequality (6.19) for $n>2r$ do not hold without the modification we have made in the previous section.

Let F be given by (6.20) with $n>2r$ and $|I|\le 2r$ and define

$$a_n^{\emptyset}(t)=\sup_{x_1}\int\prod_{j=2}^{n}d^2x_j\frac{1}{2^n}\sum_{\boldsymbol{\sigma}\in\{-1,1\}^n}\left|\frac{\sigma_1+\cdots+\sigma_n}{n}\right||\psi_n^c(t,P_n\boldsymbol{\xi})|$$

Note that $a_n^{\emptyset}\le a_n$ and, by translational invariance, the supreme over x_1 is irrelevant. As (6.21) holds with a_2^+ substituted by $a_m^{\emptyset}$, $m\le 2r$, equation (4.3) can be replaced by

$$\begin{aligned}a_n(t)\;\;\le\;\;&\frac{1}{2}\int_0^t ds\,e^{n\gamma(s,t)}\Gamma(s)\left\{\sum_{m=1}^{[n/2]}\binom{n}{m}m\,(n-m)\,a_m^{\emptyset}(s)\,a_{n-m}(s)\right.\\&\left.+\sum_{m=[n/2]+1}^{n-1}\binom{n}{m}m\,(n-m)\,a_m(s)\,a_{n-m}^{\emptyset}(s)\right\}\end{aligned} \tag{7.26}$$

where $[x]$ indicates the integer part of a real number x.

We now derive an integral equation for $a_n^\emptyset$'s similar to (4.3). We assume that the improved stability condition (2.21) holds. Applying the operation

$$\frac{1}{2^n} \sum_{\boldsymbol{\sigma} \in \{-1,1\}^n} \int \prod_{i=1}^n dx_i \sum_{j=1}^n \sigma_j \dot{v}(t, |x - x_j|)$$

in both sides of (3.16) repeating the steps employed for (4.3), with (2.8) substituted by (2.21), yields

$$a_n^\emptyset(t) \leq \frac{1}{2} \int_0^t ds\, e^{(n-\delta_n)\gamma(s,t)} \Gamma(s) \sum_{m=1}^{n-1} \binom{n}{m} m\,(n-m)\, a_m^\emptyset(s)\, a_{n-m}^\emptyset(s) \tag{7.27}$$

with $\delta_n > 1/n$ and $n = 1, \ldots, 2r$. Here, as in equation (6.21), we have used translational invariance of $\dot{v}(t, |x-y|)$ and, consequently, invariance of $\psi_n^c(\boldsymbol{\xi})$ under translation $\boldsymbol{\xi} \longrightarrow \boldsymbol{\xi}'$ of all components $x_j' = x_j + a$ by $a \in \mathbb{R}^2$.

By construction (see analogous equation (4.4)), the Mayer coefficients, regularized so that $b_{2j} = 0$ if $j \leq r$, satisfy

$$nb_n \leq \begin{cases} a_n^\emptyset/(n-1)! & \text{if} \quad n \leq 2r \\ a_n/(n-1)! & \text{if} \quad n > 2r \end{cases}$$

and, to conclude the proof, we need to show

$$1 + \sum_{j=2}^{2r} \frac{a_j^\emptyset(t)}{(j-1)!} z^{j-1} + \sum_{n=2r+1}^{\infty} \frac{a_n(t)}{(n-1)!} z^{n-1} \leq \Theta^{(2r+1)}(t,z)$$

which is equivalent to

$$C_n \geq \begin{cases} a_n^\emptyset/(n-1)! \equiv c_n^\emptyset & \text{if} \quad 1 < n \leq 2r \\ a_n/(n-1)! \equiv c_n & \text{if} \quad n > 2r \end{cases} \tag{7.28}$$

where $(C_n)_{n \geq 2}$, given by

$$\Theta^{(k)} = 1 + \sum_{n=2}^{\infty} C_n z^{n-1} \ ,$$

satisfy, in view of (7.22), the system of equations

$$\dot{C}_n = \frac{k+1}{k}(n-1)\dot{B} C_n + \frac{n\Gamma}{2} \sum_{j=1}^{n-1} C_j\ C_{n-j} \tag{7.29}$$

for $n = 2, \ldots, k = 2r+1$ and

$$\dot{C}_n = n\dot{B} C_n + \frac{n\Gamma}{2} \sum_{j=1}^{n-1} C_j\ C_{n-j} \tag{7.30}$$

for $n > k$, with initial data $C_n(t_0) = 0$ if $n > 1$ and $C_1(t) = 1$ for all $t \in [t_0, 0]$.

Comparing (4.11) with (7.22), we observe that the Lagrange multiplier L_k, $k \geq 1$, leads to a modest improvement, from n to $(k+1)(n-1)/k = n - (k-n+1)/k$, on the coefficient of the linear term C_n:

$$n - 1 < n - \frac{k-n+1}{k} \leq n - \frac{1}{n} \ . \tag{7.31}$$

On the other hand, equations (7.27) and (7.26) are equivalent to the system

$$\dot{c}_n^{\emptyset} \leq (n-\delta_n)\dot{B}c_n^{\emptyset} + \frac{n\Gamma}{2}\sum_{j=1}^{n-1} c_j^{\emptyset}\, c_{n-j}^{\emptyset}$$

for $n = 2, \ldots, 2r+1$ and

$$\dot{c}_n \leq n\dot{B}c_n + \frac{n\Gamma}{2}\sum_{j=1}^{[n/2]} c_j^{\emptyset}\, c_{n-j} + \frac{n\Gamma}{2}\sum_{j=[n/2]+1}^{n-1} c_j^{\emptyset}\, c_{n-j}$$

for $n > r+1$ with the same initial data $c_n^{\emptyset}(0) = 0$ if $1 < n \leq 2r+1$, $c_n(0) = 0$ if $n > 2r+1$ and $a_1^{\emptyset}(t) = 1$, from which, together with (7.29), (7.30) and $\delta_n < 1/n$, equations (7.28) follow.

The Mayer series for the density of the Yukawa gas, with the singular part of the norm of ψ_{2j}^c for $j = 2, \ldots, r$ removed, converges if

$$\beta < \beta_{2r+1} \qquad \text{and} \qquad |z| < \frac{\beta_{2r+1}-\beta}{\beta_{2r+1}\beta e}$$

with β_n as in (6.10). This concludes the proof of Theorem 2.3.

□

We close this section with some comments on the improved stability condition.

Remark 7.3 *The non neutrality constraint is responsible for the improvement (2.21) since (2.8) saturates if configurations with* $\sigma_1 + \cdots + \sigma_n = 0$ *are allowed. For* $n = 2$*,*

$$\dot{U}_2(t; x_1, x_2, +, +) = \dot{U}_2(t; x_1, x_2, -, -) = \dot{v}\,(t, |x_1 - x_2|) \geq 0$$

gives $\delta_2 = 2$ *and this value exceeds* $1/2$*. For* n *small,* $\dot{U}_n(t; \boldsymbol{\xi})$ *can be minimized over non neutral states* $\boldsymbol{\xi}$ *on a computer. Using the Mathematica software, the minimum for* $n = 3$

$$\begin{aligned} \min_{\boldsymbol{\xi}} \dot{U}_3(t; \boldsymbol{\xi}) &\geq 2\min_{w\geq 0}\left(h(2w) - 2h(w)\right)\dot{B} \\ &= 2\left(h(2w^*) - 2h(w^*)\right)\dot{B} = -2.1162876...\cdot \dot{B} \end{aligned}$$

is attained for a charge configuration disposed in a straight line with alternate signal separated by an equal distance $w^* = 0.4132466....$ *This gives a* $\delta_3 = 0.8837124...$ *larger than* $1/3 = 0.3333333....$

We have also found numerically the minimal configurations for non neutral quadrupoles and quintupoles. For $n = 4$ *one of the exceeded charge is expelled to infinity and the remaining charges dispose as in the* $n = 3$ *case:*

$$\min_{\boldsymbol{\xi}:\sum_j \sigma_j \neq 0} \dot{U}_4(t; \boldsymbol{\xi}) \geq 2\left(h(2w^*) - 2h(w^*)\right)\dot{B} = -2.1162876...\cdot \dot{B}$$

For $n = 5$ *the minimum is attained for a charge configuration disposed in a straight line with alternate signal and symmetric with respect to the plane intercepting the central charge perpendicular*

to the line:

$$\begin{aligned}\min_{\boldsymbol{\xi}} \dot{U}_5(t;\boldsymbol{\xi}) &\geq 2\left(h(2(w_1^*+w_2^*)) + h(2w_1^*) + 2h(w_1^*+w_2^*)\right.\\ &\qquad\left. -2h(2w_1^*+w_2^*) - 2h(w_1^*) - 2h(w_2^*)\right)\dot{B}\\ &= -4.1879447...\cdot\dot{B}\end{aligned}$$

where $w_1^* = 0.4399588...$ *and* $w_2^* = 0.2816336...$ *are the distances of the central charge to the neighbor and next neighbor charges, respectively. In both case, we have* $\delta_4 = 1.8837124... > 0.25$ *and* $\delta_5 = 0.8120552... > 0.2$.

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